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On Multivariate Polynomials of Least Deviation from Zero on the Unit Cube

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In the family of all r -variable real polynomials with total degree not exceeding μ and with maximum norm on the unit-cube not exceeding 1, any of the leading coefficients is maximum for a special product of one-variable Chebyshev polynomials of the first kind. This is a consequence of an even more general result on polynomials of least deviation from zero on the unit cube.

1. INTRODUCTION

We deal with polynomials

$$P(x) = \sum c_n x^n, \quad c_n \in \mathbb{R}, \quad (1.1)$$

in r variables $(x_1, \dots, x_r) = x \in \mathbb{R}^r$, $r \in \mathbb{N}$. Here we put

$$x^n = x_1^{n_1} \cdots x_r^{n_r} \quad \text{for } n = (n_1, \dots, n_r) \in \mathbb{N}_0^r.$$

Although our particular concern are the real polynomial spaces

$$\mathbb{P}_\mu^r := \text{span}\{x^n : |n| \leq \mu\},$$

where $\mu \in \mathbb{N}_0$, $|n| = n_1 + \cdots + n_r$ for $n \in \mathbb{N}_0^r$, we shall use even the more sophisticated real spaces

$$\Pi_m^r := \text{span}\{x^n : |n| \leq |m| \text{ or } n_i < m_i \text{ for one } i\}$$

for $m \in \mathbb{N}_0^r$.

With T_ν denoting the common Chebyshev polynomial with degree ν of the first kind we can introduce

$$T_m(x) := T_{m_1}(x_1) \cdots T_{m_r}(x_r)$$

for $m \in \mathbb{N}_0^r$ as its r -variable generalization of degree m . Note that

$$T_m(x) = c_m^* x^m + \text{lower degree terms}, \quad (1.2)$$

where

$$c_m^* = 2^{|m|-r} \quad \text{for } m > 0 \quad (1.3)$$

(i.e., $m_i > 0$ for all i).

It is well known that, with respect to

$$\begin{aligned} \|P\| &:= \max\{|P(x)| : x \in C^r\}, \\ C^r &:= [-1, 1]^r, \quad \text{unit cube,} \end{aligned}$$

the polynomial T_m is extremal in some sense within the space

$$\mathbb{P}_m^r := \text{span}\{x^n : n \leq m\}.$$

See Ehlich and Zeller [2], Sloss [4] (subspace). We are going to generalize these results for \mathbb{P}_μ^r and even for Π_m^r . Note that $\|T_m\| = 1$ and that

$$T_m \in \mathbb{P}_m^r \subset \mathbb{P}_\mu^r \subset \Pi_m^r \quad \text{for } |m| = \mu. \quad (1.4)$$

Using divided difference methods, we shall prove the following theorems:

THEOREM 1. *Let $m \in \mathbb{N}_0^r$ be fixed. Among all $P \in \Pi_m^r$ with the pivot coefficient $c_m = c_m^*$, the polynomial $P = T_m$ minimizes the maximum norm on the unit cube C^r .*

THEOREM 2. *Let $m \in \mathbb{N}_0^r$ be fixed. Among all $P \in \Pi_m^r$ with maximum norm on the unit cube C^r not exceeding one, the polynomial $P = T_m$ maximizes the pivot coefficient c_m in absolute value.*

We note that, by (1.4), the two theorems remain valid if Π_m^r is replaced either by \mathbb{P}_μ^r , $|m| = \mu$, or by \mathbb{P}_m^r .

We discuss the problem of uniqueness in Section 3. Here we should point out, however, that Theorem 2 with \mathbb{P}_μ^r instead of Π_m^r reads more detailed as

COROLLARY 3. *Let $P(x) = \sum_{|m| \leq \mu} c_m x^m$, $\|P\| \leq 1$. Then $|c_m| \leq c_m^*$ for $|m| = \mu$.*

We consider it a remarkable fact that, apart from the few cases where $m_i = 0$ for one i , all the leading coefficients c_m , $|m| = \mu$, have the same upper bound $2^{\mu-r}$ in absolute value though this bound is attained for different polynomials, compare (1.3).

2. PROOFS

Let $m \in \mathbb{N}_0^r$ be fixed and let $\tilde{\Pi}_m^r$ denote the subspace of Π_m^r where $c_m = 0$. Define

$$\mathcal{E}_\nu := \{\xi_0^{(\nu)}, \dots, \xi_\nu^{(\nu)}\}$$

for $\nu \in \mathbb{N}$ to be the set of all critical points of T_ν on the interval $[-1, 1]$ (see Rivlin [3]) and for $\nu = 0$ by $\xi_0^{(0)} := 0$, where these points are assumed to be ordered as follows:

$$\xi_0^{(\nu)} < \xi_1^{(\nu)} < \dots < \xi_\nu^{(\nu)}.$$

Now define

$$\mathcal{E}_m := \mathcal{E}_{m_1} \times \mathcal{E}_{m_2} \times \dots \times \mathcal{E}_{m_r}$$

for $m \in \mathbb{N}_0^r$. Note that \mathcal{E}_m is a subset of all critical points of T_m on the unit cube C^r and that, for

$$\tau_n = (\xi_{n_1}^{(m_1)}, \dots, \xi_{n_r}^{(m_r)}) \in \mathcal{E}_m \quad (2.1)$$

we have

$$T_m(\tau_n) = (-1)^{|m|+|n|}. \quad (2.2)$$

Now, let $P \in \Pi_m^r$ be fixed. We are going to take divided differences from P which act on nodes belonging to \mathcal{E}_m only. As can be seen from its effect to monomials, the result of the following process is independent of the order in which the several divided-difference operators are applied. The process is this: For $i = 1, 2, \dots, r$ we apply the divided-difference operator with respect to the single variable x_i which belongs to \mathcal{E}_i as its system of nodes. The result of the whole process is a polynomial $[P]_m$. Note that among all the monomials x^n which span Π_m^r , there is only one for which $[x^n]_m$ is not vanishing. This is the monomial x^m where, by usual Newton–Horner arguments, we obtain

$$[x^m]_m = 1.$$

Hence we have

$$[P]_m = c_m \quad (2.3)$$

for any $P \in \Pi_m^r$, $m \in \mathbb{N}_0^r$.

On the other hand, it can easily be seen that, as a divided difference, $[P]_m$ has the representation

$$[P]_m = \sum_{\tau \in \mathcal{E}_m} \lambda(\tau) P(\tau), \quad (2.4)$$

where for the τ_n of (2.1) we have

$$(-1)^{|m|+|n|} \lambda(\tau_n) > 0.$$

This together with (2.2) yields

$$T_m(\tau) \lambda(\tau) > 0 \quad \text{for all } \tau \in \mathcal{E}_m. \quad (2.5)$$

Now we can make the following statements. If we define

$$\Sigma: \mathcal{E}_m \rightarrow \{-1, +1\}$$

by $\Sigma(\tau) = \operatorname{sgn} \lambda(\tau)$, then Σ is an extremal signature for $\tilde{\Pi}_m^r$ in the sense of Rivlin [3]. This follows from the fact that the right-hand side of (2.4) vanishes for all $P \in \tilde{\Pi}_m^r$, which is a consequence of (2.3). On the other hand, we learn by (2.5) that Σ is associated with T_m . Hence, by Rivlin [3, Theorem 2.6], it follows that zero is a best approximation to T_m in $\tilde{\Pi}_m^r$ with respect to the maximum norm on the unit cube. This statement is equivalent to the assertion of Theorem 1.

For Theorem 2, assume $P \in \Pi_m^r$, $\|P\| \leq 1$. If $c_m = 0$, nothing needs to be proved. Now let $c_m \neq 0$. Define

$$\tilde{P} := (c_m^*/c_m) P - T_m.$$

Then $\tilde{P} \in \tilde{\Pi}_m^r$, hence by Theorem 1

$$|c_m^*/c_m| \geq |c_m^*/c_m| \cdot \|P\| = \|T_m + \tilde{P}\| \geq \|T_m\| = 1,$$

and Theorem 2 is proved.

3. UNIQUENESS

Due to an example of Buck [1], neither in Theorem 1 nor in Theorem 2 can we obtain uniqueness in the general case. This is true even, if we restrict ourselves to \mathbb{P}_μ^r , $\mu = |m|$, instead of Π_m^r .

In order to find conditions where uniqueness holds, assume again $P \in \Pi_m^r$ to be any polynomials with pivot $c_m = c_m^*$. Then

$$\tilde{P} := T_m - P \in \tilde{\Pi}_m^r.$$

Now assume

$$\|P\| = \|T_m\|.$$

Then

$$T_m(\tau) \tilde{P}(\tau) \geq 0 \quad \text{for all } \tau \in \mathcal{E}_m.$$

Besides, for \tilde{P} we can write (2.4) as

$$0 = \sum_{\tau \in \tilde{\mathcal{E}}_m} |\lambda(\tau)| T_m(\tau) \tilde{P}(\tau)$$

because of (2.5). Together this yields

$$\tilde{P}(\tau) = 0 \quad \text{for all } \tau \in \tilde{\mathcal{E}}_m. \quad (3.1)$$

Now, uniqueness of the solution of the extreme value problem of Theorem 1 would follow, if we could conclude from (3.1) that \tilde{P} is the null polynomial. This is impossible in the general case where $P \in \Pi_m^r$ and even in the case where $P \in \mathbb{P}_\mu^r$, $\mu = |m|$, as the example of Buck tells us, but possible, if we deal with \mathbb{P}_m^r instead of Π_m^r . This result has been found already by Ehlich and Zeller [2].

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